ETE3 and the Collatz Conjecture

...

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@SecureSet

Problem Statement

Start with a number

If the number is one, stop

If the number is even, divide it by two

If the number is odd, multiply by three and add one

Odd numbers become even numbers and alter their factorizations, even numbers lose a factor of two at each iteration until they become odd.

$$f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

The Big Question...

Do all numbers return to one?

Lets try a few examples...

4 -> 2 -> 1

3 -> 10 -> 5 -> 16 -> 8 -> 4 -> 2 -> 1

15 -> 46 -> 23 -> 70 -> 35 -> 106 -> 53 -> 160 -> 80 -> 40 -> 20 -> 10(!) -> 5 -> 16 -> 8 -> 4 -> 2 -> 1

27 -> 82 -> 41 -> 124 -> 62 -> 31 -> 94 -> 47 -> 142 -> 71 -> 214 -> 107 -> 322 -> 161 -> 484 -> 242 -> 121 -> 364 -> 182 -> 91 -> 274 -> 137 -> 412 -> 206 -> 103 -> 310 -> 155 -> 466 -> 233 -> 700 -> 350 -> 175 -> 526 -> 263 -> 790 -> 395 -> 1186 -> 593 -> 1780 -> 890 -> 445 -> 1336 -> 668 -> 334 -> 167 -> 502 -> 251 -> 754 -> 377 -> 1132 -> 566 -> 283 -> 850 -> 425 -> 1276 -> 638 -> 319 -> 958 -> 479 -> 1438 -> 719 -> 2158 -> 1079 -> 3238 -> 1619 -> 4858 -> 2429 -> 7288 -> 3644 -> 1822 -> 911 -> 2734 -> 1367 -> 4102 -> 2051 -> 6154 -> 3077 -> 9232 -> 4616 -> 2308 -> 1154 -> 577 -> 1732 ->866 -> 433 -> 1300 -> 650 -> 325 -> 976 -> 488 -> 244 -> 122 -> 61 -> 184 -> 92 -> 46 -> 23 -> 70 -> 35 -> 106 -> 53 -> 160 -> 80 -> 40 -> 20 -> 10 -> 5 -> 16 -> 8 -> 4 -> 2 -> 1

Stopping Time and Interesting Sequences

Stopping time is the number of steps for an input to return to one. Twenty seven has a stopping time of 111

Sequence of Stopping Times

0, 1, 7, 2, 5, 8, 16, 3, 19, 6, 14, 9, 9, 17, 17, 4, 12, 20, 20, 7, 7, 15, 15, 10, 23, 10, 111, 18, 18, 18, 106, 5, 26, 13, 13, 21, 21, 21, 34, 8, 109, 8, 29, 16, 16, 16, 104, 11, 24, 24... (A006577)

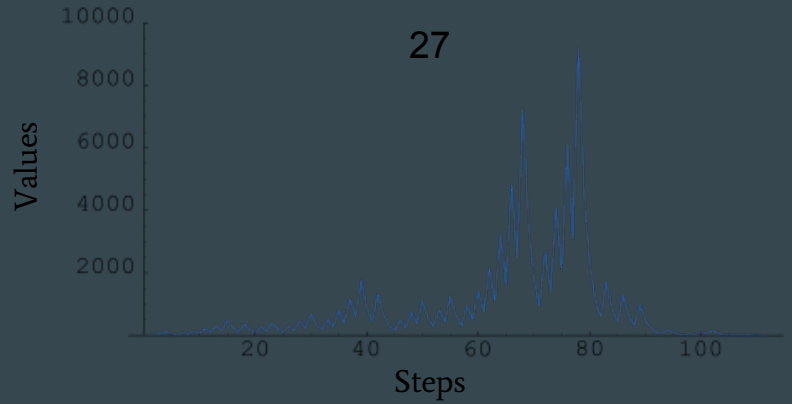
Sequence of Inputs with Largest Stopping Times

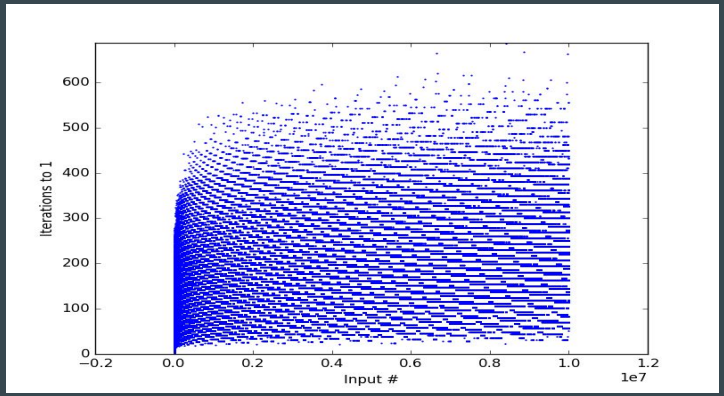
1, 2, 3, 6, 7, 9, 18, 25, 27, 54, 73, 97, 129, 171, 231, 313, 327, 649, 703, 871, 1161, 2223, 2463, 2919, 3711, 6171... (A006877)

Sequence of Inputs with Largest Peak Values

1, 2, 3, 7, 15, 27, 255, 447, 639, 703, 1819, 4255, 4591, 9663, 20895, 26623, 31911, 60975, 77671, 113383, 138367, 159487, 270271, 665215, 704511... (A006884)

Powers of two have a stop time equal to their exponent





Convergent Return Paths and Delay Classes

There exist consecutive inputs which have the same stopping time, how do they converge?

12 -> 6 -> 3 -> 10 -> 5 -> 16 -> 8 -> 4 -> 2 -> 1

13 -> 40 -> 20 -> 10 -> 5 -> 16 -> 8 -> 4 -> 2 -> 1

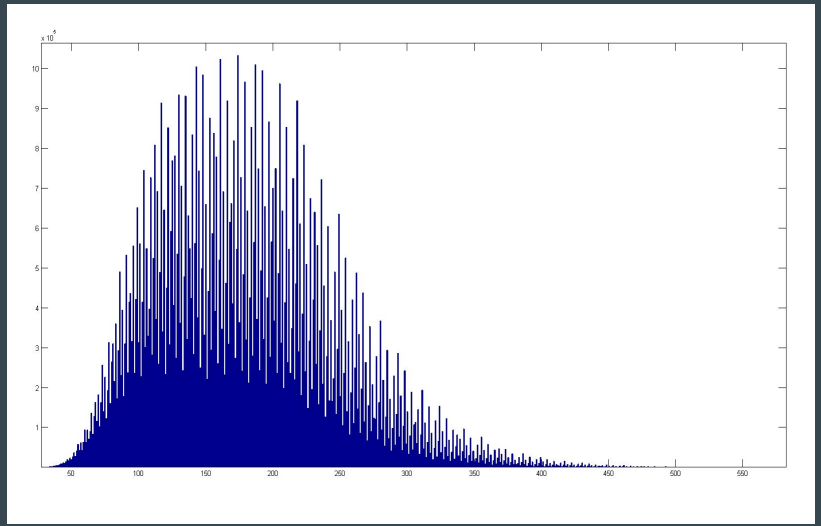
Inputs with the same stopping time are said to be in the same Delay Class, the lowest such number is known as the Class Record.

12 and 13 are both part of Delay Class 9, but 12 is also the Class Record.

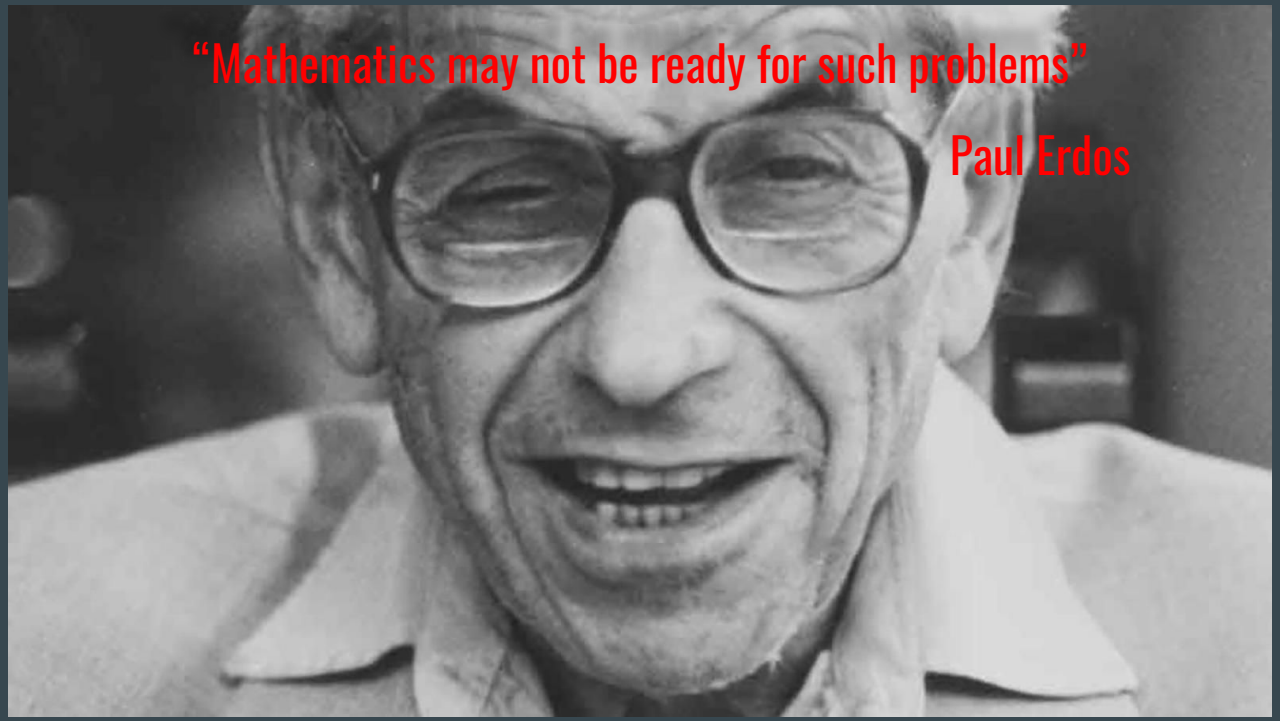
All members of a Delay Class eventually converge, and typically early on.

The largest class record thus far is for class 2258, that number is 279,731,455,495,736,617.

First 100,000 numbers grouped into Delay Classes



But Why



The Naive Implementation

Considerations

- Python
  - Big number support
  - Library availability
  - Generally efficient
- Statistic Logging
  - Steps, Peaks, Orbits
  - Consecutive Runs
  - CSV Output
- Inefficiencies
  - Global Interpreter Lock
  - Duplicative
  - About 30 minutes to compute up to 10M

```
def collatz(k):
 peakValue = 1
 returnSteps = 0
 orbitCount = 0
 growthFactor = 0
 origin = k
 loflag = 0
 hiflag = 0
 while k!= 1:
  if origin < k:
   hiflag = 1
   loflag = 0
  if origin > k:
   loflag = 1
   hiflag = 0
  if k > peakValue:
   peakValue = k
  if k%2 == 0:
   k /= 2
   returnSteps+=1
   if hiflag and (k < origin):
    orbitCount+=1
  else:
   k = k*3 + 1
   returnSteps+=1
   if loflag and (k > origin):
    orbitCount+=1
 growthFactor = peakValue/origin
 data = (origin, returnSteps, peak
 return data
```

```
maximum return steps is 685 and occurs at 8400511
maximum peak value is 60342610919632.0 and occurs at 6631675
maximum orbit count is 113 and occurs at 8546945
maximum growth factor is 9099150.80573641 and occurs at 6631675
maximum step run is 65 at input 5772712
maximum peak run is 16 at input 8965036
maximum orbit run is 2 at input 1
[terra @ versions] $
```

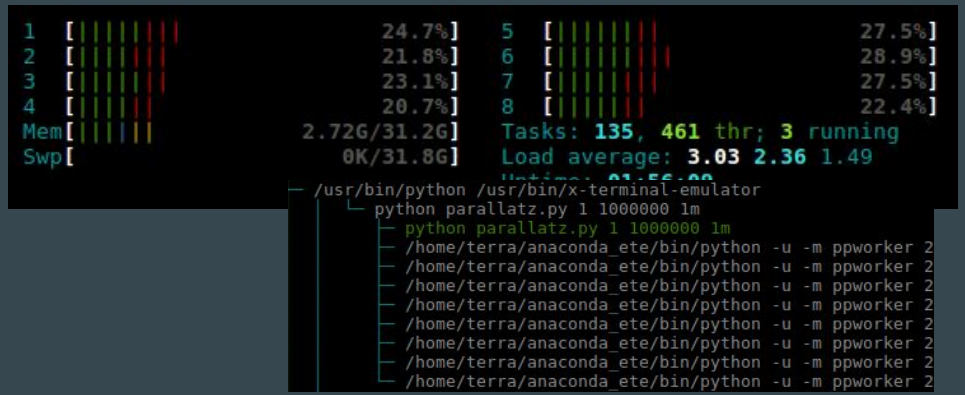
```
  1  [|                                    2.6%]     5  [|||                                  4.7%]
  2  [|||||||||||||||||||||||||||||||||||99.3%]     6  [||                                   4.0%]
  3  [|                                    1.4%]     7  [||                                   2.0%]
  4  [|                                    0.7%]     8  [|                                    1.3%]
  Mem[||||||                          2.35G/31.2G]   Tasks: 121, 486 thr; 3 running
  Swp[                                  0K/31.8G]   Load average: 2.04 2.03 1.44
                                                    Uptime: 00:29:42

  PID USER      PRI  NI  VIRT   RES   SHR S CPU% MEM%   TIME+  Command
 4114 terra      20   0  680M 56624 34024 S  5.4  0.2  0:52.15 ├─ /usr/bin/pytho
 4281 terra      20   0  293M  273M  5668 R 100.  0.9 10:33.70 │  │  └─ python3
 4123 terra      20   0  680M 56624 34024 S  0.0  0.2  0:01.28 │  ├─ /usr/bin/py
 4122 terra      20   0  680M 56624 34024 S  0.0  0.2  0:00.00 │  └─ /usr/bin/py
 3172 terra      20   0  235M 25584 11788 S  0.0  0.1  0:00.13 ├─ /usr/bin/py
 3182 terra      20   0  235M 25584 11788 S  0.0  0.1  0:00.00 │  ├─ /usr/bin
 3181 terra      20   0  235M 25584 11788 S  0.0  0.1  0:00.00 │  └─ /usr/bin
```

```
(18082, 185, 48904.0, 26, 0.010231169118460347, 2.7045680787523505, 0.001437894038270103)
(18081, 92, 54244, 13, 0.005088214147447597, 3.000055306675516, 0.0007189867817045517)
(18082, 185, 48904.0, 26, 0.010231169118460347, 2.7045680787523505, 0.001437894038270103)
(18083, 185, 81376.0, 23, 0.010230603329093623, 4.500138251396339, 0.0012719128463197478)
(18084, 185, 77272.0, 24, 0.010230037602300375, 4.2729484627294845, 0.0013271400132714001)
(18085, 185, 77272.0, 25, 0.010229471938070224, 4.272712192424661, 0.0013823610727121925)
(18086, 185, 77272.0, 28, 0.01022890633639279, 4.272475948247263, 0.0015481587968594493)
(18087, 92, 154528.0, 1, 0.005086526234311937, 8.543594847127771, 5.52883286338254e-05)
(18088, 22, 18088, 0, 0.0012162759840778417, 1.0, 0.0)
(18089, 185, 695464.0, 13, 0.010227209906573055, 38.44679086737796, 0.0007186688042456742)
(18090, 22, 27136.0, 2, 0.0012161415146489773, 1.5000552791597568, 0.0001105583195135434)
```

The Parallel Naive Implementation

Considerations

- pp library for threaded jobs
- tqdm module for loading bar
- CPU Utilization +~100%
- Writing 60GB+ files can crash things
- Still recomputing values :(




```
job_server = pp.Server()

def statlog():
 data_list = []
 lim1 = batchStart
 lim2 = batchStart + 8
 for i in tqdm(range(batchStart,batchEnd)):
  job_list = [job_server.submit(collatz, (i,)) for
  data_list.extend([job() for job in job_list])
  lim1+=8
  lim2+=8
 open(fileName, "w").write("\n".join(data_list))
 job_server.print_stats()
statlog()
```

```
[terra @ versions] $ python parallatz.py 1 10000 10k
100%|| 10000/10000 [00:12<00:00, 786.01it/s]
Job execution statistics:
 job count | % of all jobs | job time sum | time per job | job server
     80000 |        100.00 |      17.4876 |     0.000219 | local
Time elapsed since server creation 12.7268002033
0 active tasks, 8 cores
```

```
[terra @ versions] $ python parallatz.py 1 100000 100k
100%|| 100000/100000 [02:07<00:00, 785.10it/s]
Job execution statistics:
 job count | % of all jobs | job time sum | time per job | job server
    800000 |        100.00 |     179.1408 |     0.000224 | local
Time elapsed since server creation 127.406756878
0 active tasks, 8 cores
```

```
[terra @ versions] $ python parallatz.py 1 1000000 1m
100%|| 1000000/1000000 [21:51<00:00, 762.60it/s]
Job execution statistics:
 job count | % of all jobs | job time sum | time per job | job server
   8000000 |        100.00 |     1865.0076 |     0.000233 | local
Time elapsed since server creation 1311.58688903
0 active tasks, 8 cores
```

Refactoring for Dynamic Growth

The Inverse Problem Statement

In order to explore the Collatz problem with a tree structure it makes sense to grow from existing data, rather than to trace paths to a root.

$$R(n) = \begin{cases} \{2n\} & \text{if } n \equiv 0, 1, 2, 3, 5 \\ \{2n, (n-1)/3\} & \text{if } n \equiv 4 \end{cases} \pmod 6.$$

The inverse function doesn't stop on a condition, so depth of the tree will be the practical limit.

There are also multiple outputs on the alternative condition, so a decision must be made about the order of processing, I decided to use a breadth-first approach with a queue.

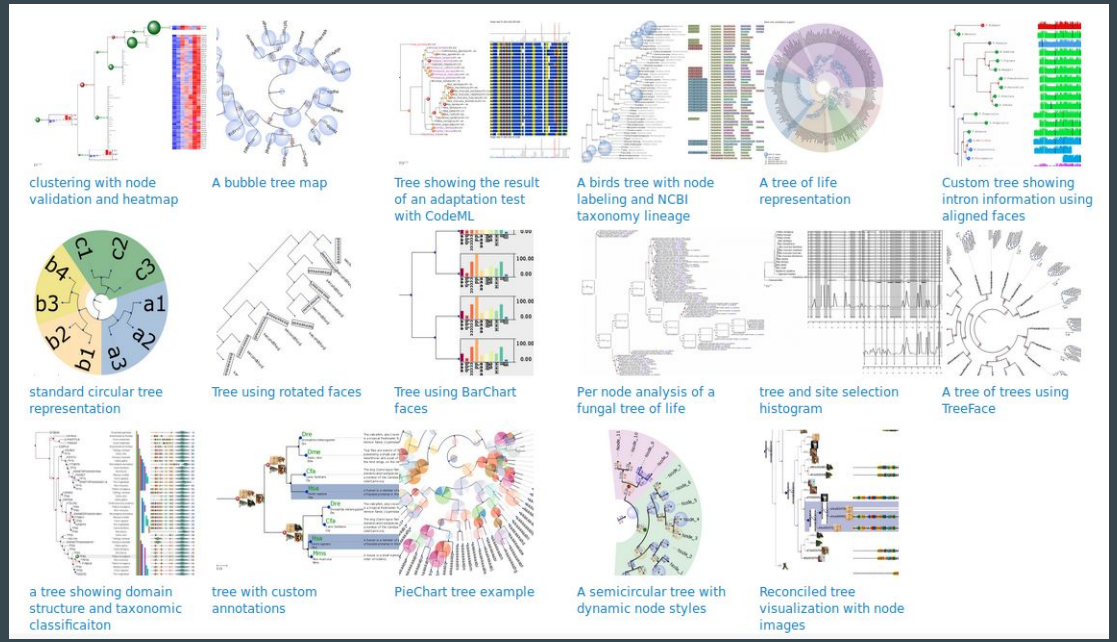
```
depth = int(sys.argv[1])

class Node:
    def __init__(self, value):
        self.value = int(value)
    def norm(node): #next normal collatz node
        self.norm = node
    def inv1(node): #first inverse collatz node
        self.inv1 = node
    def inv2(node): #second inverse collatz node
        self.inv2 = node
    def dist(value): #distance from root
        self.dist = value

class colTree:
    def __init__(self, maxDepth):
        self.root = Node(1)
        self.root.dist = 0
        self.nodeQueue = deque([self.root])
        self.visited = [1]
        self.newickIndex = 2
        self.commaFlag = 0
        self.currentDist = 0

    def grow(self):
        while (nodeQueue):
            self.rCol(nodeQueue.popleft())
```

ETE3 - The (Python) Environment for Tree Exploration

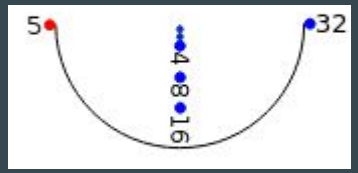
Features

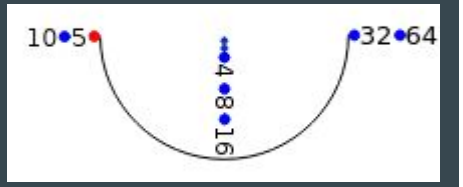
- Dynamic, Programmable, Powerful
- Newick Format
- Rendering Capabilities
- Phylogenetic-specific functions

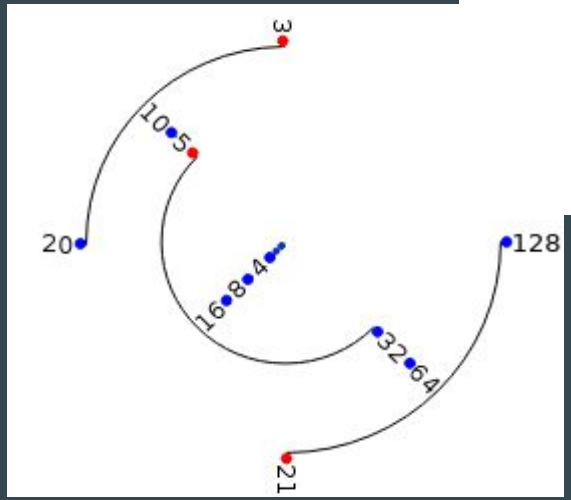
```
from ete3 import Tree
tree = Tree('((A,B), D);')

print tree
#       /-A
#    /-|
# --|   \-B
#    \-D

A = tree & "A"
A.up.show()
```

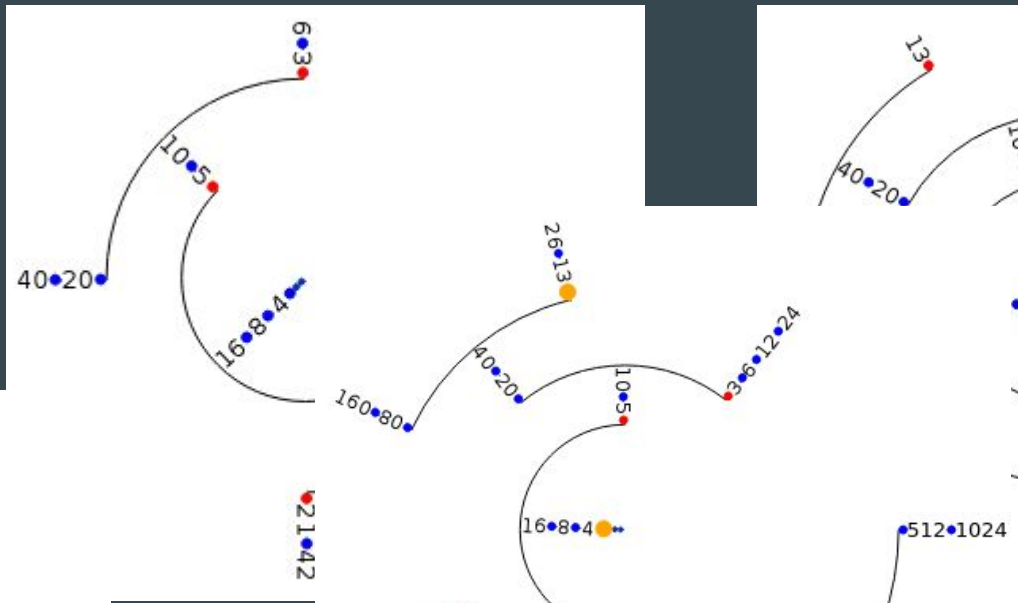


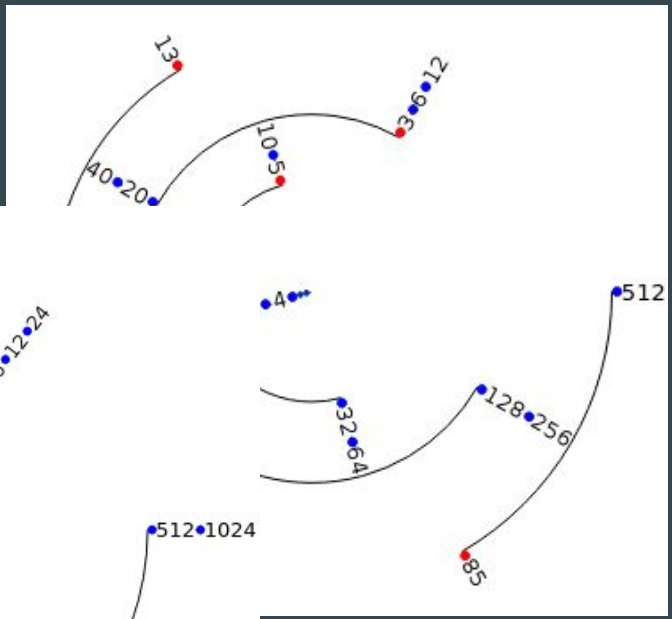
Graphing Time



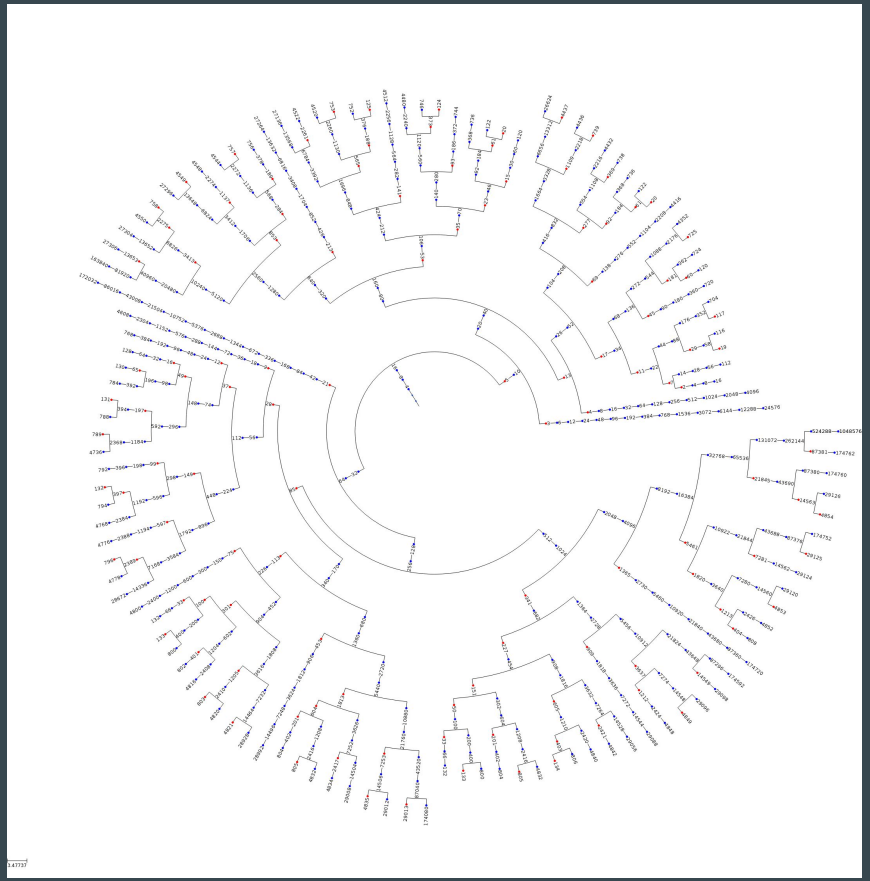


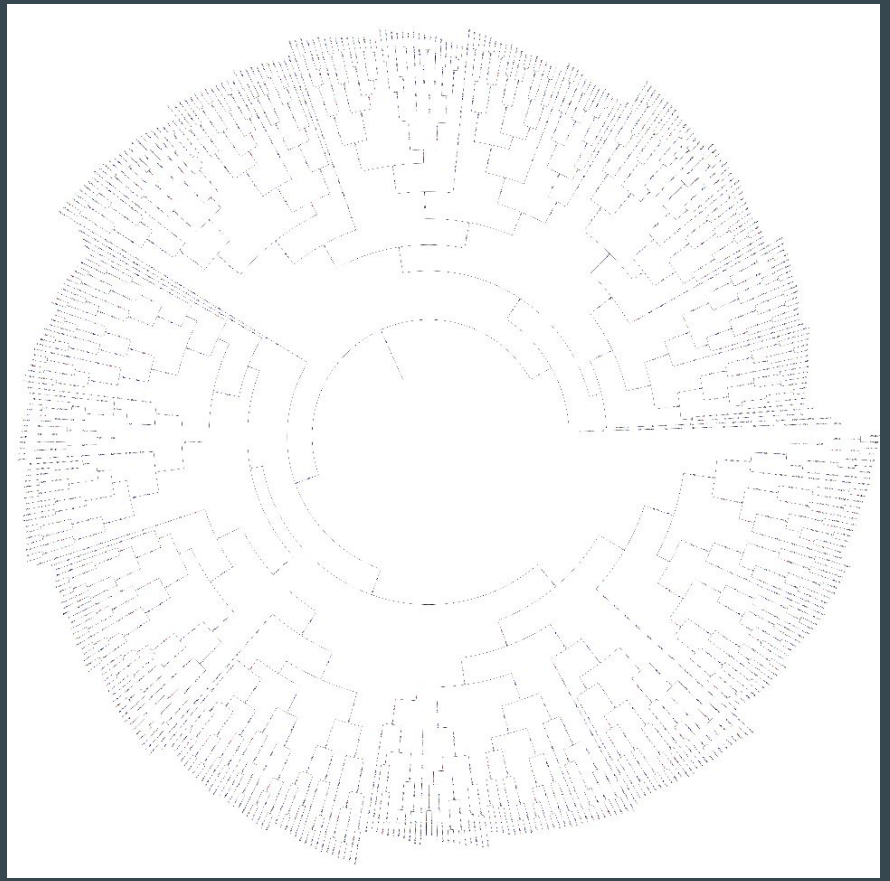


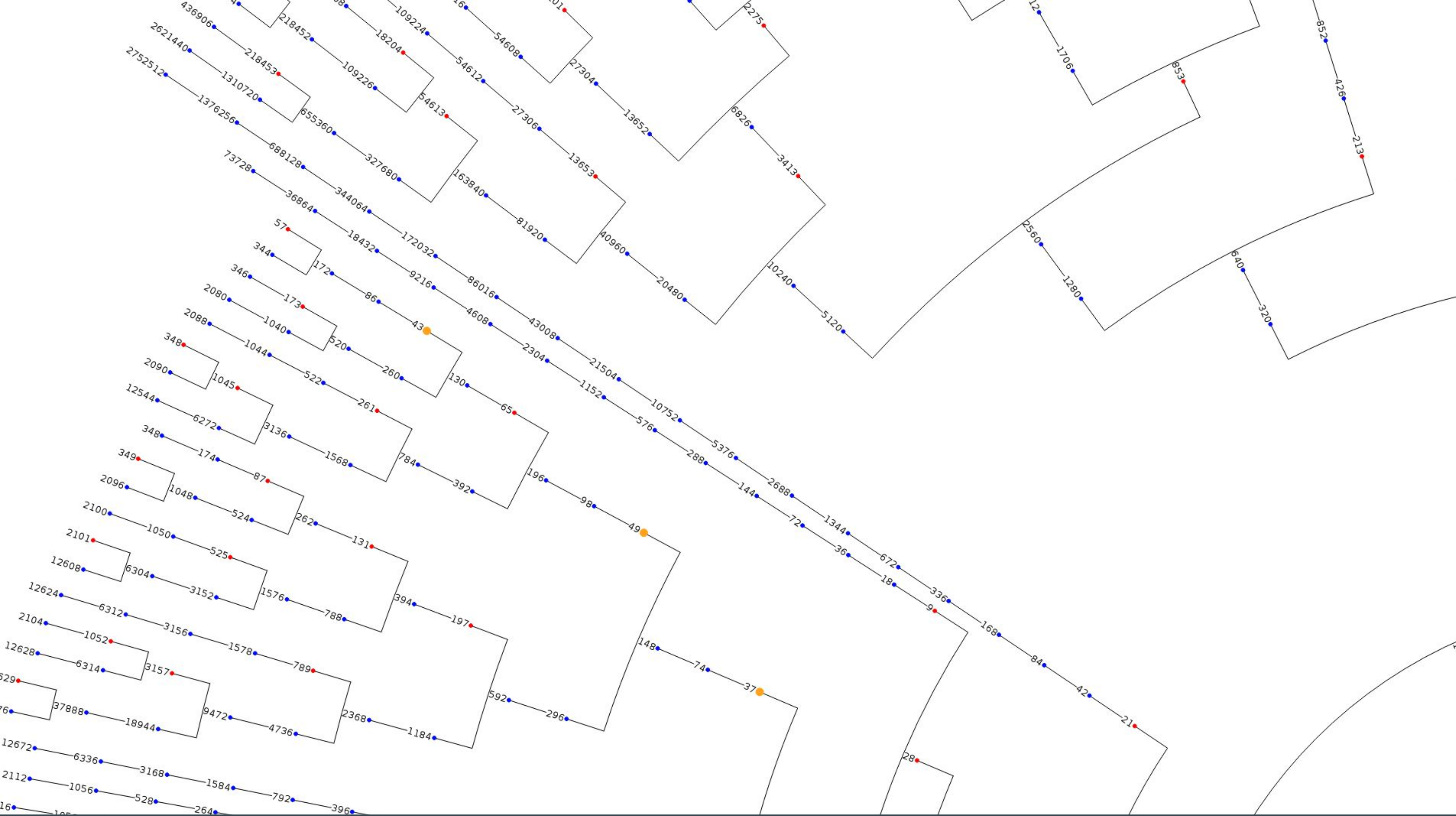




I Like Big Graphs







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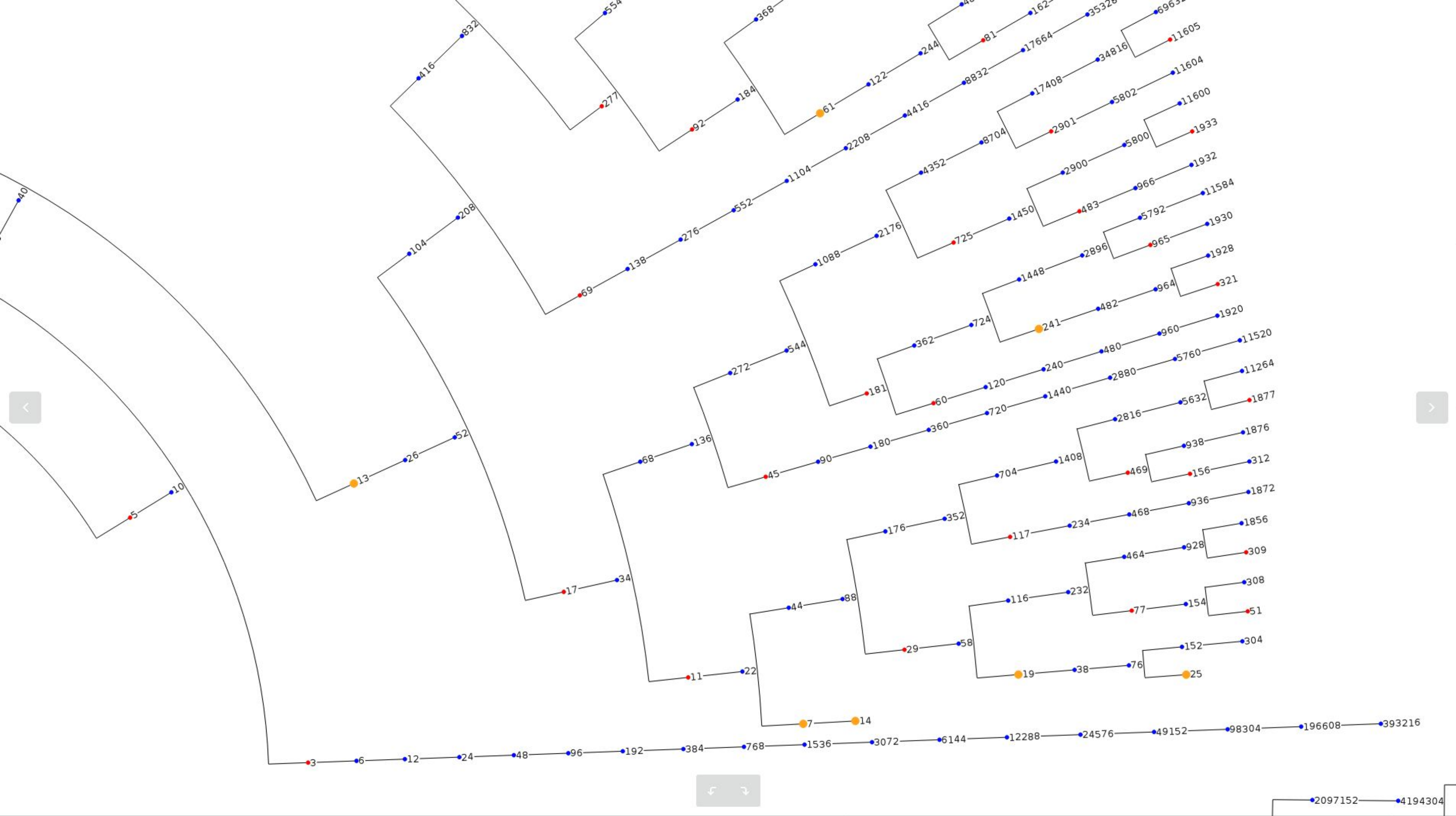
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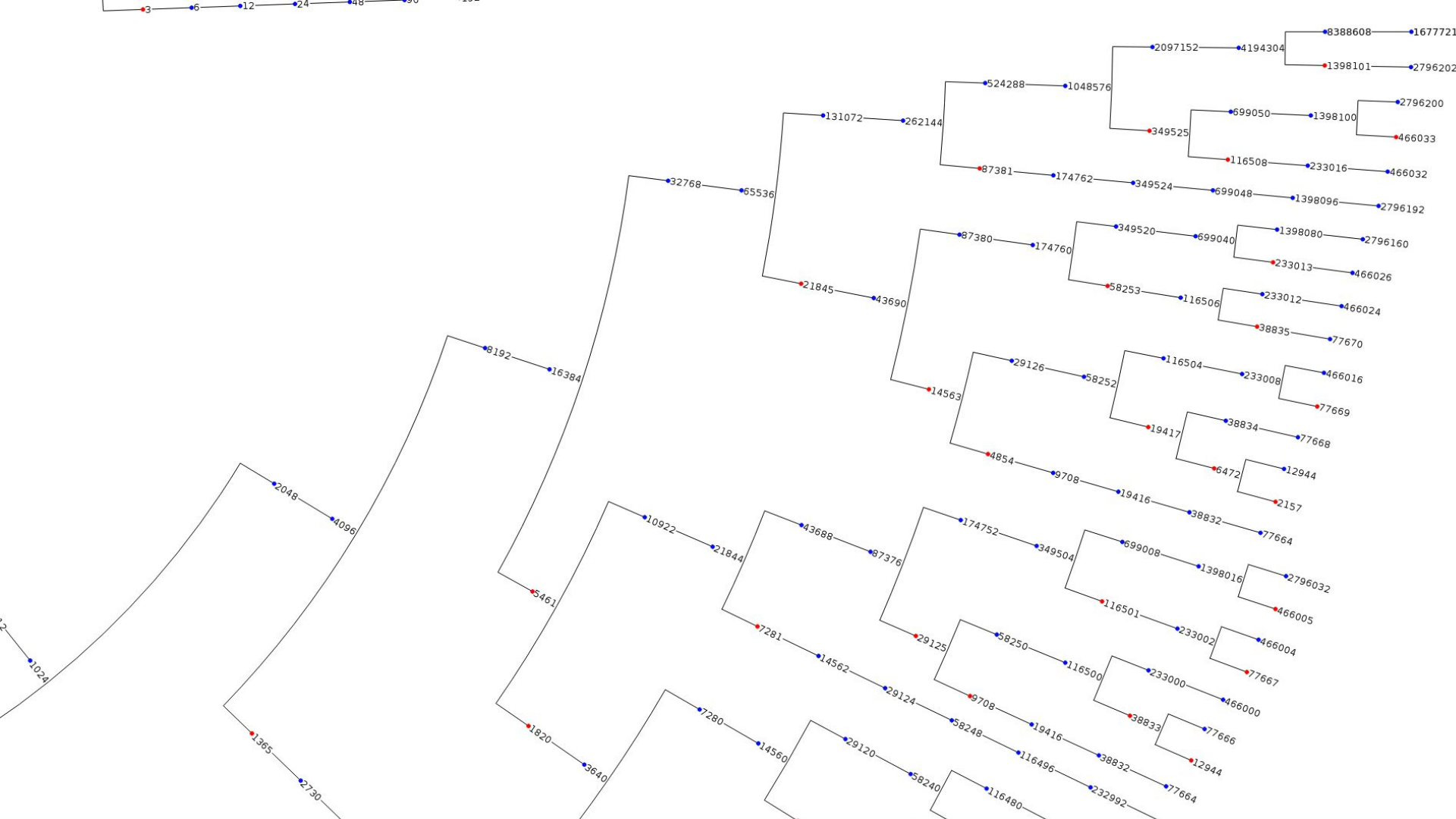
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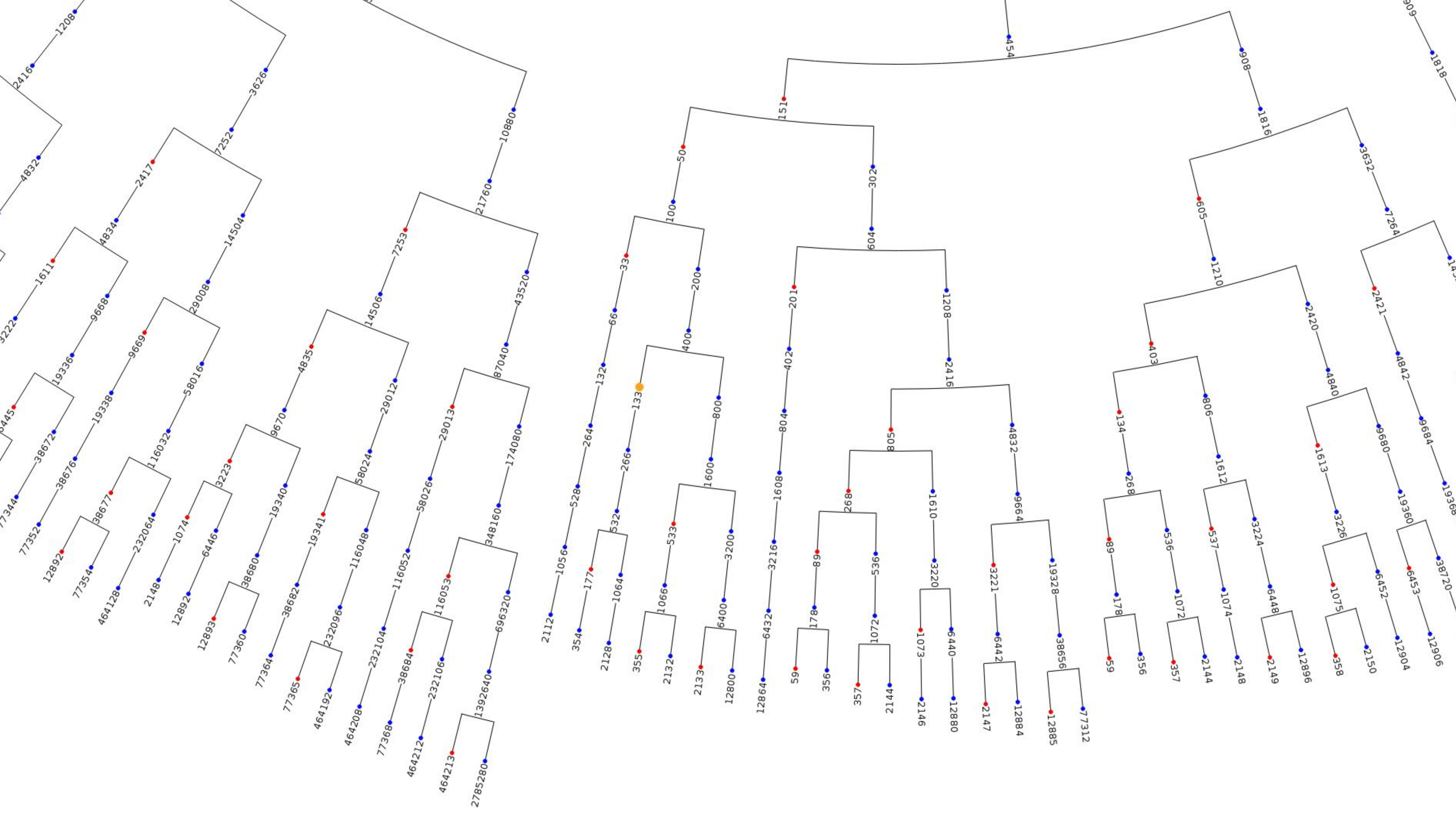

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